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Approximation of Random Functions

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1. INTRODUCTION

In this paper we study the problem of approximating two continuous functions simultaneously by one approximating function. Our motivation is the following. Let $f_1(x)$ and $f_2(x)$ be continuous real-valued functions, each defined on $a \leqslant x \leqslant b$, occurring with probabilities w_1 and w_2, respectively, $w_1 + w_2 = 1$. The function F is an approximating function chosen before f_1 and f_2 are observed. The error is a random variable which assumes the value $\|f_1 - F\|$ with probability w_1 and $\|f_2 - F\|$ with probability w_2. (Here $\|\cdot\|$ denotes a suitably chosen norm.) Our goal is to choose F from a given approximating family so as to minimize the expected value of the error, i.e., choose F to minimize

$$w_1\|f_1 - F\| + w_2\|f_2 - F\|.$$

The direction of our investigation is a search for conditions which distinguish the minimizing F from other elements of the approximating family. When the polynomials of degree n or less are used as the approximating family and the norm is chosen to be the Chebychev norm, we find a necessary condition for the minimizing polynomial. If f_1 and f_2 are ordered, the necessary condition is also sufficient.

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Study of a related problem has been reported on in the approximation literature by Bacopoulos and others, see [1–10, 12–16].

2. A Necessary Condition for Best Approximations

For a bounded real-valued function, $g(x)$, defined on the compact real interval $[a, b]$, we define the norm of g by $\| g \| = \sup_{a \leq x \leq b} | g(x) |$.

Our approximating family, $\mathscr{F}$, is a family of continuous real-valued functions defined on $[a, b]$, and the two functions to be approximated, $f_1(x)$ and $f_2(x)$, are given continuous real-valued functions also defined on $[a, b]$. In addition, the weight functions, $w_1(x)$ and $w_2(x)$, are nonnegative real-valued continuous functions such that $w_1(x) + w_2(x) = 1$ for each x in $[a, b]$. A function F_0 in $\mathscr{F}$ is said to be a best l_1-approximant to f_1 and f_2 if

$$\| w_1(f_1 - F_0)\| + \| w_2(f_2 - F_0)\| = \inf_{F \in \mathscr{F}} [\| w_1(f_1 - F)\| + \| w_2(f_2 - F)\|].$$

In this section we give a necessary condition for F_0 to be a best l_1-approximant when the approximating family is a family of polynomials (see Theorem 2.3).

For each F in $\mathscr{F}$ the error, $\| w_1(f_1 - F)\| + \| w_2(f_2 - F)\|$, is the sum of two errors, namely $\| w_1(f_1 - F)\|$ and $\| w_2(f_2 - F)\|$. Sometimes it will be necessary to show the dependence of these two errors on both F and f_i $(i = 1, 2)$ and sometimes it will suffice to show the dependence only on f_i $(i = 1, 2)$; so with a slight abuse of notation we write, for each F in $\mathscr{F}$,

$$E_i = E(F, f_i) = \| w_i(f_i - F)\| \qquad (i = 1, 2).$$

Also defined are an "upper error function," $M(F, x)$, and a "lower error function," $m(F, x)$: For each F in $\mathscr{F}$ and for each x in $[a, b]$,

$$M(F, x) = \min_{i=1,2} \{f_i(x) + E_i/w_i(x)\},$$

and

$$m(F, x) = \max_{i=1,2} \{f_i(x) - E_i/w_i(x)\}.$$

If there exists x_0 in $[a, b]$ and i (either $i = 1$ or $i = 2$) such that $w_i(x_0) = 0$, we employ the convention, $f_i(x_0) \pm E_i/w_i(x_0) = \pm\infty$. However, the requirement $w_1(x_0) + w_2(x_0) = 1$ guarantees that $M(F, x_0)$ and $m(F, x_0)$ are both finite.

The proof of the following lemma is straightforward and is omitted.

Lemma 2.1. *For each F in $\mathscr{F}$ and for each x in $[a, b]$ one has*

$$m(F, x) \leq F(x) \leq M(F, x).$$

LEMMA 2.2. *Let F and G belong to $\mathscr{F}$ such that $m(F, x) < G(x) < M(F, x)$ for all x in $[a, b]$. Then one has*

$$\| w_1(f_1 - G)\| + \| w_2(f_2 - G)\| < \| w_1(f_1 - F)\| + \| w_2(f_2 - F)\| .$$

Proof. For each i $(i = 1, 2)$ and for all x in $[a, b]$ one has, from the hypothesis,

$$f_i(x) - E(F, f_i)/w_i(x) < G(x) < f_i(x) + E(F, f_i)/w_i(x).$$

This means that either $E(F, f_1) > 0$ or $E(F, f_2) > 0$. Thus for at least one i $(i = 1, 2)$

$$-E(F, f_i) < w_i(x)[f_i(x) - G(x)] < E(F, f_i)$$

for all x in $[a, b]$; and for the other i,

$$-E(F, f_i) \leqslant w_i(x)[f_i(x) - G(x)] \leqslant E(F, f_i)$$

for all x in $[a, b]$. Since all the functions are continuous, one concludes that for at least one i, $\| w_i(f_i - G)\| < E(F, f_i)$ and for the other i, $\| w_i(f_i - G)\| \leqslant E(F, f_i)$. Thus,

$$\begin{aligned} \| w_1(f_1 - G)\| + \| w_2(f_2 - G)\| &< E(F, f_1) + E(F, f_2) \\ &= \| w_1(f_1 - F)\| + \|(w_2(f_2 - F)\| . \end{aligned}$$

Remark. The above proof shows that if $m(F, x) \leqslant G(x) \leqslant M(F, x)$ for all x in $[a, b]$ then for one i, $\| w_i(f_i - G)\| < E(F, f_i)$ and for the other i, $\| w_i(f_i - G)\| \leqslant E(F, f_i)$, which is a stronger conclusion than the conclusion of the lemma as stated.

Our aim now is to show that if F in $\mathscr{F}$ is a best l_1-approximant to f_1 and f_2 then it is a best approximant in another sense, to two functions related to f_1 and f_2. Known necessary conditions for best approximation in this other sense, can then be translated to necessary conditions for best l_1-approximation. The next definition is made for this purpose.

DEFINITION 2.1. A function F_0 in $\mathscr{F}$ is said to be a best l_∞-approximant to two continuous real-valued functions, $g_1(x)$, $g_2(x)$ defined on $[a, b]$ if

$$\max\{\| g_1 - F_0 \|, \| g_2 - F_0 \|\} = \inf_{F \in \mathscr{F}} \max\{\| g_1 - F \|, \| g_2 - F \|\}.$$

The following theorem establishes a connection between the l_1-problem and the l_∞-problem.

THEOREM 2.1. *Let F belong to $\mathscr{F}$, and define $g_1(x) = m(F, x) + \| M(F) - m(F)\|$ and $g_2(x) = M(F, x) - \| M(F) - m(F)\|$, for all x in $[a, b]$. If F is a best l_1-approximant to f_1 and f_2 then F is a best l_∞-approximant to g_1 and g_2.*

Proof. Lemma 2.2 guarantees that for each G in $\mathscr{F}$ there exists an x_0 in $[a, b]$ such that either $G(x_0) \geqslant M(F, x_0)$, or $G(X_0) \leqslant m(F, X_0)$. Thus, either $\| g_2 - G \| \geqslant \| M(F) - m(F)\|$ or $\| g_1 - G \| \geqslant \| M(F) - m(F)\|$. Thus, $\inf_{G \in \mathscr{F}} \max \{\| g_1 - G \|, \| g_2 - G \|\} \geqslant \| M(F) - m(F)\|$. On the other hand, using Lemma 2.1 and the fact that $g_2 \leqslant g_1$ gives

$$\| g_2 - F \| \leqslant \| M(F) - m(F)\| \qquad \text{and} \qquad \| g_1 - F \| \leqslant \| M(F) - m(F)\|.$$

It follows that

$$\inf_{G \in \mathscr{F}} \max\{\| g_1 - G \|, \| g_2 - G \|\} = \max\{\| g_1 - F \|, \| g_2 - F \|\},$$

i.e., F is a best l_∞-approximant to g_1 and g_2.

Remark. The above proof does not use the fact that F is a best l_1-approximant to f_1 and f_2. It uses the fact that there exists no G in $\mathscr{F}$ such that both $E(G, f_1) < E(F, f_1)$ and $E(G, f_2) < E(F, f_2)$. There are in general many such elements F in $\mathscr{F}$. Theorem 2.1 then allows one to state necessary conditions for these F's in terms of known necessary conditions for the best l_∞-approximants to the corresponding g_1's and g_2's.

In [12], e.g., conditions are given which are necessary for F to be a best l_∞-approximant to g_1 and g_2. We repeat them here, but to do so requires the following two definitions [12].

DEFINITION 2.2. A point x_0 in $[a, b]$ is called an l_∞-straddle point for F (in $\mathscr{F}$) with respect to $g_1(x)$ and $g_2(x)$($g_2(x) \leqslant g_1(x)$ for all x in $[a, b]$) if

$$\max\{\| g_1 - F \|, \| g_2 - F \|\} = g_1(x_0) - F(x_0) = F(x_0) - g_2(x_0).$$

(We observe that such an F is necessarily a best l_∞-approximant to g_1 and g_2.)

DEFINITION 2.3. An l_∞-approximant, F (in $\mathscr{F}$), to $g_1(x)$ and $g_2(x)$ ($g_2(x) \leqslant g_1(x)$ for all x in [a, b]) is said to l_∞-alternate n times on $[a, b]$ if there exist $n + 1$ points, x_i $(0 \leqslant i \leqslant n)$, $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ such that at least one of the following two conditions hold.

(1) For each even i in $\{0, 1, \ldots, n\}$ and for each odd j in $\{0, 1, \ldots, n\}$ both $g_1(x_i) - F(x_i)$ and $F(x_j) - g_2(x_j)$ assume the value $\max \{\| g_1 - F \|, \| g_2 - F \|\}$, or

(2) the same is true for each odd i and each even j in $\{0, 1, \ldots, n\}$.

The following theorem deals with the case when the approximating family $\mathscr{F}$ is P_n, the polynomials of degree n or less.

THEOREM 2.2 (see, e.g., [12]). *The element F in P_n is a best l_∞-approximant to the continuous real functions g_1, g_2 ($g_2(x) \leqslant g_1(x)$ for all x in $[a, b]$) if and only if F has an l_∞-stranddle point with respect to g_1 and g_2 or F l_∞-alternates $n + 1$ times.*

To translate this characterization to the l_1 problem we need two definitions.

DEFINITION 2.4. A point x_0 in $[a, b]$ is said to be an l_1-straddle point for F (in $\mathscr{F}$) with respect to f_1 and f_2 if $m(F, x_0) = M(F, x_0)$.

DEFINITION 2.5. An l_1-approximant F (in $\mathscr{F}$) to f_1 and f_2 is said to l_1-alternate n times on $[a, b]$ if there exist $n + 1$ points $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ such that at least one of the following two conditions holds.

(1) For each even i in $\{0, 1,..., n\}$ and for each odd j in $\{0, 1,..., n\}$ both $M(F, x_i) = F(x_i)$ and $m(F, x_j) = F(x_j)$, or

(2) the same is true for each odd i and each even j.

The following theorem is the main theorem of this section.

THEOREM 2.3. *If the element F in P_n is a best l_1-approximant to the two real continuous functions $f_1(x), f_2(x)$, $a \leqslant x \leqslant b$, ($f_1$ and f_2 are not necessarily ordered) then either F has an l_1-straddle point or F l_1-alternates n times.*

Proof. The proof follows in a straightforward manner from Theorems 2.1 and 2.2.

The following example shows that F may alternate without being a best l_1-approximant.

EXAMPLE. Let $[a, b] = [-2, 2]$, $w(x) = w_2(x) = \frac{1}{2}$,

$$\begin{aligned} f_1(x) &= x + 2, && -2 \leqslant x \leqslant 0, \\ &= -x + 2, && 0 < x \leqslant 2, \\ f_2(x) &= x + 3, && -2 \leqslant x \leqslant 0, \\ &= (-\tfrac{3}{2})\, x + 3, && 0 < x \leqslant 2 \end{aligned}$$

and $\mathscr{F} = P_1$. It is easily checked that $F(x) = (-1/4)\, x + 7/4$ l_1-alternates two times on $[-2, 2]$. However, $F(x)$ is not a best l_1-approximant. Indeed, the best l_1-approximants to f_1 and f_2 are exactly the polynomials of the form $F(x) = K$, $1 \leqslant K \leqslant \frac{3}{2}$.

Remark. It can be shown that if a given F in P_n, l_1-alternates n times on $[a, b]$ then for each $G \in P_n$, $G \neq F$ either $E(G, f_1) > E(F, f_1)$ or $E(G, f_2) >$

$E(F, f_2)$. This observation could form the basis of a computational technique for computing best l_1-approximants from P_n. In [15] a computational technique is discussed which does not use this observation but which could be modified to do so.

Since l_1-alternation is not sufficient to ensure best l_1-approximation it would be convenient to know how the definition of alternation must be altered to obtain a sufficient condition. The authors believe that such a discovery would lead to a more efficient computational technique than that discussed in [15].

3. l_1-Approximation of Ordered Functions

We consider now the problem of approximating in the l_1 sense, two real continuous functions, $f_1(x)$ and $f_2(x)$, which are pointwise ordered: $f_2(x) \leqslant f_1(x)$ for all x in $[a, b]$. The approximating family, $\mathscr{F}$, is assumed to be a linear family of real continuous functions on $[a, b]$ and for ease of exposition we assume the weight functions $w_1(x)$ and $w_2(x)$ are identically equal to one on $[a, b]$ ($w_1(x) = w_2(x) = 1$). Our goal is to show that when $f_1(x) \leqslant f_1(x)$ for all x in $[a, b]$ and when $\mathscr{F} = P_n$, there exists a theorem which gives necessary and sufficient conditions for best l_1-approximation in terms of alternation and straddle points.

Lemma 3.1. *Let $f_2(x) \leqslant f_1(x)$ for all x in $[a, b]$ and F in $\mathscr{F}$ be a best l_1-approximant to f_1 and f_2. Then there exist an x_1 and x_2 in $[a, b]$ such that $F(x_1) = f_1(x_1) - E(F, f_1)$ and $F(x_2) = f_2(x) + E(F, f_2)$.*

Proof. We prove that x_1 exists. That x_2 exists can be shown using similar ideas. The proof is by contraposition. Assume that there does not exist such an x_1, i.e., $F(x) > f_1(x) - E(F, f_1)$ for all x in $[a, b]$. Since both $F(x)$ and $f_1(x)$ are continuous on $[a, b]$, there exists x_0 in $[a, b]$ such that $F(x_0) = f_1(x_0) + E(F, f_1)$. Further, $F(x) \leqslant f_2(x) + E(F, f_2)$ for all x, so in particular, $F(x_0) \leqslant f_2(x_0) + E(F, f_2)$. Thus $f_1(x_0) + E(F, f_1) \leqslant f_2(x_0) + E(F, f_2)$, or $f_1(x_0) - f_2(x_0) \leqslant E(F, f_2) - E(F, f_1)$. Since the left-hand side is nonnegative, one concludes that $E(F, f_1) \leqslant E(F, f_2)$.

Thus

$$f_1(x) - E(F, f_1) \geqslant f_2(x) - E(F, f_2)$$

for all x in $[a, b]$ and using the original assumption, one has, $F(x) > f_2(x) - E(F, f_2)$ for all x in $[a, b]$. Recalling the definition of $m(x)$, given in Section 2, we have shown that $F(x) > m(x)$ for all x in $[a, b]$. Now defining $c = \frac{1}{2} \min_{a \leqslant x \leqslant b} (F(x) - m(x))$, and noting that c is positive, one has $M(x) > F(x) - c > m(x)$ for all x in $[a, b]$. Using Lemma 2.2, and the fact that $\mathscr{F}$ is a linear family, one concludes that $F(x) - x$ is a better l_1-approximant to f_1 and f_2 than is $F(x)$.

LEMMA 3.2. *Let $f_2(x) \le f_1(x)$ for all x in $[a, b]$ and F in $\mathcal{F}$ be a best l_1-approximant to f_1 and f_2, and define $c = \frac{1}{2}(E(F, f_1) - E(F, f_2))$. Then $F + c$ is also a best l_1-approximant to f_1 and f_2 and further $E(F + c, f_1) = E(F + c, f_2)$.*

Proof. We assume first that $E(F, f_1) > E(F, f_2)$. Using the previous lemma one concludes that $E(F + c, f_2) = E(F, f_2) + c$. We show next that $E(F + c, f_1) = E(F, f_1) - c$. On one hand one has $F(x) + c \le f_2(x) + E(F, f_2) + c = f_2(x) + E(F, f_1) - c \le f_1(x) + E(F, f_1) - c$ for all x in $[a, b]$. And on the other hand, $F(x) + c \ge f_1(x) - E(F, f_1) + c = f_1(x) - (E(F, f_1) - c)$ for all x on $[a, b]$ with equality holding for at least one x by the previous lemma. Hence $E(F + c, f_1) = E(F, f_1) - c$. Combining these results gives $E(F + c, f_1) + E(F + c, f_2) = E(F, f_1) + E(F, f_2)$ Thus $F + c$ is a best l_1-approximant. It is clear that $E(F + c, f_1) = E(F + c, f_2)$. The case, $E(F, f_1) < E(F, f_2)$ is treated similarly. The case $c = 0$ is trivial.

Remark 3.1. If one assumes, e.g., that $\mathcal{F}$ is a finite-dimensional linear family it is easy to prove that f_1 and f_2 have a best l_1-approximant. We note that Lemma 3.2 therefore guarantees the existence of a best l_1-approximant to f_1 and f_2 ($f_2 \le f_1$) with equal errors.

LEMMA 3.3. *Let F in $\mathcal{F}$ be a best l_1-approximant to f_1 and f_2 with $E(F, f_1) = E(F, f_2)$. Then F is a best l_∞-approximant to f_1 and f_2.*

Proof. The proof follows from the fact that for every G in $\mathcal{F}$

$$\begin{aligned} 2 \max\{\|f_1 - G\|, \|f_2 - G\|\} &\ge E(G, f_1) + E(G, f_2) \\ &\ge E(F, f_1) + E(F, f_2) = 2 \max\{\|f_1 - F\|, \\ &\qquad \|f_2 - F\|\}. \end{aligned}$$

Remark 3.2. The above two lemmas show that when $\mathcal{F}$ is a linear family and $f_2(x) \le f_1(x)$ for all x in $[a, b]$ then every best l_1-approximant is a translate of some best l_∞-approximant.

Remark 3.3. We note that the proof of Lemma 3.3 does not depend upon the linearity of $\mathcal{F}$ nor the ordering of f_1 and f_2.

LEMMA 3.4. *Let f_1, f_2 be given with $f_2(x) \le f_1(x)$ for all x in $[a, b]$. Assume F in $\mathcal{F}$ is a best l_∞-approximant to f_1, f_2 and let*

$$c_2 = [\max_{x \in [a,b]} (f_1(x) - F(x)) + \min_{x \in [a,b]} (f_1(x) - F(x))]/2$$

and

$$c_1 = [\max_{x \in [a,b]} (f_2(x) - F(x)) + \min_{x \in [a,b]} (f_2(x) - F(x))]/2.$$

Then the following are true.

(a) $c_2 \geqslant 0$, *and* $c_1 \leqslant 0$.

(b) *If* $c \in [c_1, c_2]$, *then* $F + c$ *is a best* l_1*-approximant to* f_1, f_2. *In particular,* F *is a best* l_1*-approximant to* f_1, f_2.

(c) *If* $c \in (-\infty, c_1) \cup (c_2, +\infty)$, *then* $F + c$ *is not a best* l_1*-approximant to* f_1, f_2.

Proof. It is convenient to show first that F is a best l_1-approximant to f_1 and f_2. To see this we note that $E(F, f_1) = E(F, f_2)$ (which can be easily verified). Now let G in $\mathscr{F}$ be a best l_1-approximant to f_1 and f_2 with the property that $E(G, f_1) = E(G, f_2)$. Then one has

$$\begin{aligned} E(F, f_1) + E(F, f_2) &= 2 \max\{\|F - f_1\|, \|F - f_2\|\} \\ &\leqslant 2 \max\{\|G - f_1\|, \|G - f_2\|\} \\ &= E(G, f_1) + E(G, f_2), \end{aligned}$$

which means that F is also a best l_1-approximant to f_1 and f_2.

Since F is a best l_1-approximant, Lemma 3.1 may be employed yielding $\max_{x \in [a,b]} (f_1(x) - F(x)) = E(F, f_1)$, $\min_{x \in [a,b]} (f_1(x) - F(x)) \geqslant -E(F, f_1)$ and thus $c_2 \geqslant 0$. Similarly $c_1 \leqslant 0$ holds. We prove next part (b) and (c) of the lemma.

Let $c \in [0, c_2]$. From Lemma 3.1, it follows that $E(F + c, f_2) = E(F, f_2) + c$. We show that $E(F + c, f_1) = E(F, f_1) - c$.

First observe that $F(x) + c \geqslant f_1(x) - E(F, f_1) + c = f_1(x) - (E(F, f_1) - c)$ is valid on $[a, b]$, with equality for some value of x, due to Lemma 3.1. Second, note that $c \leqslant c_2 = [E(F, f_1) + \min_{x \in [a,b]} (f_1(x) - F(x)]/2$ implies that $c \leqslant [E(F, f_1) + (f_1(x) - F(x))]/2$ for all $x \in [a, b]$. Rearranging this last inequality yields $F(x) + c \leqslant f_1(x) + E(F, f_1) - c$ on $[a, b]$. Therefore $E(F + c, f_1) = E(F, f_1) - c$ and we obtain $E(F + c, f_1) + E(f + c, f_2) = E(F, f_1) + E(F, f_2)$.

Next let $c \in (c_2, +\infty)$. Again by Lemma 3.1, we have $E(F + c, f_2) = E(F, f_2) + c$. However with $c > c_2$, a rearrangement of this inequality yields, $E(F + c, f_1) > E(F, f_1) - c$. Thus $E(F + c, f_1) + E(F + c, f_2) > E(F, f_1) + E(F, f_2)$, and therefore $F + c$ is not a best l_1-approximant to f_1, f_2.

A similar argument for $c \in [c_1, 0]$ and $c \in (-\infty, c_1)$ completes the proof.

We conclude Section 3 with an alternation theorem for a best l_1-approximant of two ordered functions.

THEOREM 3.1. *Let* $f_1(x)$ *and* $f_2(x)$ *be real-valued continuous functions with* $f_2(x) \leqslant f_1(x)$ *for all* x *in* $[a, b]$. *Let the approximating family* $\mathscr{F}$ *be the polynomials of degree* n *or less, and let* F *belong to* $\mathscr{F}$. *Then* F *is a best* l_1*-approximant to* f_1 *and* f_2 *if and only if at least one of the following three conditions holds.*

(1) *There exist points x_k ($0 \le k \le n$) such that $a \le x_0 < x_1 < \cdots < x_n \le b$ and such that for each even i in $\{0, 1,..., n\}$ and for each odd j in $\{0, 1,..., n\}$ both $F(x_i) = f_2(x_i) + E(F, f_2)$ and $F(x_j) = f_1(x_j) - E(F, f_1)$;*

(2) *condition (1) holds for each odd i and each even j;*

(3) *there exists an x_0 in $[a, b]$ such that*

$$f_1(x_0) - E(F, f_1) = f_2(x_0) + E(F, f_2).$$

(We think of the phenomena of conditions (1) and (2) as alternation phenomena; e.g., if condition (1) holds we say that $F(x)$ alternates between $f_2(x) + E(F, f_2)$ and $f_1(x) - E(F, f_1)$.)

Proof. Case 1. Assume $E(F, f_1) = E(F, f_2)$. If F is a best l_1-approximant then F is a best l_∞-approximant and the proof follows from Theorem 2.2. On the other hand, if one of the three conditions holds, then F is a best l_∞-approximant by Theorem 2.2 and hence a best l_1-approximant.

Case 2. Assume $E(F, f_1) > E(F, f_2)$. (The case $E(F, f_1) < E(F, f_2)$ can be argued in a similar manner.) Choose the real number c such that $F + c$ is a best l_1-approximant to f_1 and f_2 and $E(F + c, f_1) = E(F + c, f_2)$. From the proof of Lemma 3.3 one has that $E(F + c, f_1) = E(F, f_1) - c$ and $E(F + c, f_2) = E(F, f_2) + c$. This observation allows one to reduce Case 2 to Case 1 where $F + c$ plays the role of F in Case 1.

The following example demonstrates that the ordering assumption, $f_2 \le f_1$, in Theorem 3.1 is necessary.

EXAMPLE. (For simplicity we describe the example in lieu of a lengthy constructive presentation.) Let $\mathscr{F} = P_1$ and $[a, b] = [0, 1]$. One can easily construct two nonordered functions f_1, f_2 such that: (a) $F = 0$ is a best l_1-approximation from P_1, (b) F is a best Chebyshev approximation to both f_1 and f_2 from P_1, (c) alternation of $F(x)$ between $f_2(x) + E(F, f_2)$ and $f_2(x) - E(F, f_2)$ occurs in $[0, \frac{1}{2})$, (d) alternation of $F(x)$ between $f_1(x) + E(F, f_1)$ and $f_1(x) - E(F, f_1)$ occurs in $(\frac{1}{2}, 1]$, and yet (e) F does not alternate twice in the l_1 sense between either $f_1(x) - E(F, f_1)$ and $f_2(x) + E(F, f_2)$ or between $f_2(x) - E(F, f_2)$ and $f_1(x) + E(F, f_2)$. Note, however, that F will alternate twice in the l_1 sense between $M(F, x)$ and $m(F, x)$.

4. COMMON ALTERNATION POINTS

In this section we investigate further the l_1-approximation problem as discussed in Section 2. Specifically, we are not assuming that the functions f_1 and f_2 are ordered and we are not assuming that the weight functions w_1 and w_2 are both identically one. We do specialize the problem by assuming that the approximating family, $\mathscr{F}$, is linear. The real continuous functions $\phi_1(x)$, $\phi_2(x)$,..., $\phi_n(x)$ $a \leqslant x \leqslant b$, are given and assumed to be linearly independent over the real numbers. The family $\mathscr{F}$ consists of exactly those functions of the form $a_1\phi_1(x) + \cdots + a_n\phi_n(x)$ where a_i is real, $1 \leqslant i \leqslant n$. Letting A denote the vector $(a_1, ..., a_n)$ in E^n (n-dimensional Euclidean space), we denote (with slight abuse of notation) the general element of $\mathscr{F}$ by $F = F(A) = F(A, x) = a_1\phi_1(x) + \cdots + a_n\phi_n(x)$. Elements of $\mathscr{F}$ can then be represented by their corresponding vectors A. The two functions f_1 and f_2 will in general have many best l_1-approximants; we write

$$R = \{A \in E^n: \| F(A) - f_1 \| + \| F(A) - f_2 \| = \inf_{F \in \mathscr{F}} [\| F - f_1 \| + \| F - f_2 \|]\}.$$

The set R in E^n can be thought of as representing the best l_1-approximants to f_1 and f_2. Standard arguments show that R is compact, convex, and non-empty. In particular, if $F(A_1, x)$ and $F(A_2, x)$ are both best l_1-approximants to f_1 and f_2 then so is $F(A_3, x)$ where $A_3 = \lambda A_1 + (1 - \lambda) A_2$, $0 \leqslant \lambda \leqslant 1$. We use the notation $E(A, f) = E(F(A), f)$ in what follows.

LEMMA 4.1. *Let $F(A_1, x)$, $F(A_2, x)$ be best l_1-approximants to f_1, f_2 from $\mathscr{F}$. If $A_3 = \lambda A_1 + (1 - \lambda) A_2$, for $0 \leqslant \lambda \leqslant 1$, then $E(A_3, f_i) = \lambda E(A_1, f_i) + (1 - \lambda) E(A_2, f_i)$, $1 \leqslant i \leqslant 2$.*

Proof. Since $F(A_3, x)$ is a best l_1 —approximation.

$$\begin{aligned} E(A_3, f_1) + E(A_3, f_2) &= \lambda(E(A_1, f_1) + E(A_1, f_2)) + (1 - \lambda)(E(A_2, f_1) + E(A_2, f_2)) \\ &= [\lambda E(A_1, f_1) + (1 - \lambda) E(A_2, f_1)] \\ &\quad + [\lambda E(A_1, f_2) + (1 - \lambda) E(A_2, f_2)]. \end{aligned}$$

But for each i, $1 \leqslant i \leqslant 2$,

$$\begin{aligned} E(A_3, f_i) &= \| w_i(f_i - F(A_3, \cdot) \| \\ &= \| w_i[f_i - (\lambda F(A_1, \cdot) + (1 - \lambda) F(A_2, \cdot))] \| \\ &\leqslant \lambda E(A_1, f_i) + (1 - \lambda) E(A_2, f_i). \end{aligned}$$

Combining these remarks, we obtain $E(A_3, f_i) = \lambda E(A_1, f_i) + (1 - \lambda) E(A_2, f_i)$, $1 \leqslant i \leqslant 2$.

LEMMA 4.2 *Let $F(A_1, x)$, $F(A_2, x)$ be best l_1-approximants from $\mathscr{F}$ to f_1, f_2. Let $A_3 = \lambda A_1 + (1 - \lambda) A_2$, $0 < \lambda < 1$. Then*

(a) $F(A_3, x_0) = f_i(x_0) - E(A_3, f_i)/w_i(x_0)$ *if and only if*
$F(A_1, x_0) = f_i(x_0) - E(A_1, f_i)/w_i(x_0)$ *and*
$F(A_2, x_0) = f_i(x_0) - E(A_2, f_i)/w_i(x_0)$, $1 \leqslant i \leqslant 2$, $x_0 \in [a, b]$ and

(b) *a similar statement holds with a plus sign.*

Proof. We prove part (a). Since $w_i(x_0) = 0$ implies that all three expressions above are minus infinity, we may assume $w_i(x_0) \neq 0$.

Suppose $F(A_3, x_0) = f_i(x_0) - E(A_3, f_i)/w_i(x_0)$. Using Lemma 4.1, we have $\lambda E(A_1, f_i)/w_i(x_0) + (1 - \lambda) E(A_2, f_i)/w_i(x_0) = E(A_3, f_i)/w_i(x_0) = f_i(x_0) - F(A_3, x_0) = \lambda(f_i(x_0) - F(A_1, x_0)) + (1 - \lambda)(f_i(x_0) - F(A_2, x_0)) \leqslant \lambda E(A_1, f_i)/w_i(x_0) + (1 - \lambda) E(A_2, f_i)/w_i(x_0)$. Since the first and last terms are the same, equality holds throughout. We obtain

$$F(A_j, x_0) = f_i(x_0) - E(A_j, f_i)/w_i(x_0) \qquad \text{for } j = 1 \text{ and } j = 2.$$

On the other hand, suppose $F(A_j, x_0) = f_i(x_0) - E(A_j, f_i)/w_i(x_0)$ for $j = 1$ and 2. Then $f_i(x_0) - F(A_3, x_0) = \lambda(f_i(x_0) - F(A_1, x_0)) + (1 - \lambda)(f_i(x_0) - F(A_2, x_0)) = \lambda E(A_1, f_i)/w_i(x_0) + (1 - \lambda) E(A_2, f_i)/w_i(x_0) = E(A_3, f_i)/w_i(x_0)$ using Lemma 4.1 Thus we have $F(A_3, x_0) = f_i(x_0) - E(A_3, f_i)/w_i(x_0)$, $i = 1, 2$.

We turn next to the main result of this section. Recall that the set R defined earlier consists of all those parameters A in E^n which yield a best l_1-approximants. We shall say that a point $A \in R$ is an interior point of R if A is a strict convex combination of the boundary points of every line segment in R containing A. We also introduce the following terminology. A set of points $a \leqslant x_0 < x_1 < x_2 < \cdots < x_n \leqslant b$ will be called an alternation set for A if $F(A, x)$ alternates between $M(F(A, x), x)$ and $m(F(A, x), x)$ on $\{x_i\}_{i=0}^n$ in the sense of Definition 2.5. Further, a point x_0 in $[a, b]$ will be called an extreme point for A if either $F(A, x_0) = M(F(A, x_0), x_0)$ or $F(A, x_0) = m(F(A, x_0), x_0)$.

THEOREM 4.1. *Let f_1, f_2 be given and let R be the parameter set of best l_1-approximants from $\mathscr{F}$ where $\mathscr{F}$ is a linear family as described above. Assume A_3 is an interior point of R.*

(a) *If* $\text{ext}(A_3)$ *is a set of extreme points for A_3, then* $\text{ext}(A_3)$ *is a set of extreme points for every A in R.*

(b) *If* $\text{alt}(A_3)$ *is a set of alternation points for A_3, then* $\text{alt}(A_3)$ *is a set of alternation points for every A in R.*

(c) *If x_0 is an l_1-straddle point for A_3, then x_0 is an l_1-straddle point for every A in R.*

Proof. We prove part (b). Parts (a) and (c) are shown in a similar manner. Let A be any element of R and consider the line segment in R determined by A and A_3. Call A_1, A_2 the boundary points of this line segment. Suppose that x_k is in $\text{alt}(A_3)$. Then we have $F(A_3, x_k) = f_i(x_k) \pm E(A_3, f_i)/w_i(x_k)$ for some choice of i, $1 \leqslant i \leqslant 2$, and some choice of $\pm$. Applying Lemma 4.2 twice, it follows that $F(A, x_k) = f_i(x_k) \pm E(A, f_i)/w_i(x_k)$ for the same choice of i and the same choice of $\pm$. Since this is true for every x_k in $\text{alt}(A_3)$, $\text{alt}(A_3)$ is an alternation set for A.

Remark. Let $\mathscr{F} = P_n$ and let A_3 be in the interior of the set R. Observe that if $F(A_3, x)$ alternates in the l_1 sense, then Theorem 4.1 guarantees that every best l_1-approximant alternates at least $n + 1$ times.

We close Section 4 with an example which illustrates Theorem 4.1.

EXAMPLE. Let $[a, b] = [-1, 1]$, $w_1(x) = w_2(x) = \frac{1}{2}$,

$$\mathscr{F} = P_0, f_1(x) = x^2,$$

$$\begin{aligned} f_2(x) &= x + \tfrac{7}{4}, & -1 \leqslant x \leqslant -\tfrac{1}{2}, \\ &= -x + \tfrac{3}{4}, & -\tfrac{1}{2} < x \leqslant \tfrac{1}{2}, \\ &= x - \tfrac{1}{4}, & \tfrac{1}{2} < x \leqslant 1. \end{aligned}$$

It is easily checked that $R = [\frac{1}{2}, \frac{3}{4}]$, and the error $\rho = \frac{5}{8}$. For every $A \in \operatorname{int}(R) = (\frac{1}{2}, \frac{3}{4})$, $\operatorname{alt}(A) = \{-\frac{1}{2}, 0\}$ is an alternation set for A. In fact $\{-\frac{1}{2}, 0\}$ are the only extreme points for $A \in (\frac{1}{2}, \frac{3}{4})$. As predicted by Theorem 4.1, $\{-\frac{1}{2}, 0\}$ is also an alternation set for the boundary parameters $A_1 = \frac{1}{2}$ and $A_2 = \frac{3}{4}$. However, for the boundary parameters additional extreme points may exist. A simple diagram shows that in fact $\{-\frac{1}{2}, 0, -1, 1\}$ are extreme points for A_1 and $\{-\frac{1}{2}, 0, \frac{1}{2}\}$ are extreme points for A_2.

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